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Superconvergence of nonconforming finite element penalty scheme for Stokes problem using L^2 projection method*

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Abstract A modified penalty scheme is discussed for solving the Stokes problem with the Crouzeix-Raviart type nonconforming linear triangular finite element. By the L^2 projection method, the superconvergence results for the velocity and pressure are obtained with a penalty parameter larger than that of the classical penalty scheme. The numerical experiments are carried out to confirm the theoretical results.

Key words superconvergence, Crouzeix-Raviart type nonconforming finite element, penalty scheme, L^2 projection method

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1 Introduction

Consider the following incompressible Stokes problem:

$$-\Delta u + \nabla p = f \quad \text{in} \quad \Omega, \tag{1}$$

$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega, \tag{2}$$

$$u = 0 \quad \text{on} \quad \partial\Omega, \tag{3}$$

where $u = (u^1, u^2)$, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with the Lipschitz boundary $\partial\Omega$.

As we know, problem (1)–(3) is difficult to be numerically solved because the pressure p does not appear in the incompressibility equation (2). A simple way to overcome this difficulty is to add a pressure dependent perturbation to the incompressible constrain $\operatorname{div} u = 0$[1–6], in which the penalty method is the simplest and fundamental one. It has been shown in [7] that for a penalty parameter $\varepsilon > 0$, the difference between the solution of the Stokes problem and the penalty finite element approximation can be estimated to be of order $O(h^{s-1} + \varepsilon)$, where h denotes the mesh size, s is the order of complete polynomials contained in the approximating space. In this case, ε must be sufficiently small to obtain the accuracy approximation, which sometimes may result in the stiffness matrix in a bad condition. To circumvent or ameliorate the deficiency, [8] proposed a modified penalty method by the linear combination of two

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solutions obtained from the classical penalty method and proved that the approximation was more accurate than the classical penalty method for conforming elements. At the same time, the modified penalty method with a larger penalty parameter can achieve the same accuracy as the classical method with a smaller penalty parameter. Therefore, it is easier to solve the stiffness matrix generated by the modified penalty method. Later, [9] applied this method to nonconforming elements and derived the same convergence results as in [8].

Moreover, the superconvergence for finite element solutions has been an active research area in numerical analysis. In recent years, there are some excellent studies on the superconvergence of mixed finite element approximation to the stationary Stokes problem[10–19]. [10–13] derived the superconvergence for conforming elements. [14] and [15] derived the superconvergence for nonconforming constrained $Q_1^{\rm rot}$ rectangular elements[20] with the integral identity technique and interpolation postprocessing operators. In addition, [16] established a general superconvergence result for conforming mixed element approximations of the Stokes problem by the L^2 projection method proposed and analyzed in [21]. The basic idea of the L^2 projection method is to construct a new finite element approximation in a finite-dimensional space corresponding to a coarse mesh, and the difference in the mesh sizes can be used to achieve a better convergence rate after the post-processing procedure. Later, [17] extended the superconvergence analysis of [16] to the nonconforming case. Furthermore, the L^2 projection method was used in [18] and [19] to establish a superconvergence result for the stabilized finite element method.

In this paper, we investigate the superconvergence of the modified nonconforming finite element penalty scheme for the Stokes problem with the L^2 projection method. The nonconforming finite element space consists of the Crouzeix-Raviart type nonconforming linear triangular element for the velocity and the piecewise constants for the pressure. Using this space, [9] only obtained the convergence results for the modified penalty scheme. [17] derived the superconvergence by the L^2 projection method, but only the theoretical analysis of the mixed element scheme was presented without numerical experiments. As a first attempt, we combine the modified penalty method with the L^2 projection method to get the superconvergence and overcome the above difficulty in the numerical simulation of the incompressible Stokes problem. The error estimates of the discrete energy norm for velocity and the L^2 norm for pressure with order $O(h^k)\,(1<k<2)$ are obtained for the modified penalty scheme, respectively, in which the penalty parameter only needs to be chosen to be of order $O(h)$ instead of $O(h^k)\,(1<k<2)$ in the classical penalty method. At the same time, to verify the performance of the present method, we carry out a numerical example. Moreover, we point out that the results derived herein are also valid for the nonconforming finite element space satisfying some conditions (see Remark 3.2 below).

The remainder of this paper is organized as follows: In the next section, we present the modified nonconforming finite element penalty method for the Stokes problem (1)–(3). In Section 3, the superconvergence results both for velocity and pressure are obtained by the L^2 projection method. In Section 4, we present some numerical results to illustrate the validity of the present theoretical analysis.

We use the standard notation for the Sobolev spaces $H^m(\Omega)$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$ and $H^m(K)$ with norm $\|\cdot\|_{m,K}$ and semi-norm $|\cdot|_{m,K}$, where $m\geqslant 0$ is a real number. Let $\|\cdot\|_0$ and $\|\cdot\|_{0,K}$ be the $L^2(\Omega)$ norm and the $L^2(K)$ norm, respectively. Besides, let P_k be the space consisting of piecewise polynomials of degree k, where $k\geqslant 0$ is an integer. Throughout the paper, C denotes a positive constant independent of the mesh parameter h.

2 Modified penalty scheme and some preliminary results

Let

$$X=(H_0^1(\Omega))^2,$$

$$M = L_0^2(\Omega) = \Big\{q, q \in L^2(\Omega), \int_\Omega q\mathrm{d}x\mathrm{d}y = 0\Big\}.$$

Then, the weak form of (1)–(3) is to find $(u,p) \in X \times M$ such that

$$a(u,v) - b(v,p) = (f,v), \quad \forall v \in X, \tag{4}$$

$$b(u,q) = 0, \quad \forall q \in M, \tag{5}$$

where

$$a(u,v) = \int_\Omega \nabla u \nabla v \mathrm{d}x\mathrm{d}y,$$

$$b(v,q) = \int_\Omega q \operatorname{div} v \mathrm{d}x\mathrm{d}y, \quad \forall u,v \in X, \quad \forall q \in M.$$

The spaces X and M satisfy the Babuška-Brezzi (B-B) condition[7], i.e., there exists a constant $\beta_0 > 0$ such that

$$\sup_{v \in X} \frac{b(v,q)}{\|v\|_1} \geqslant \beta_0 \|q\|_0, \quad \forall q \in M.$$

Thus, it is easy to show that (4)–(5) has a unique solution.

The classical penalty method applied to (1)–(3) is to approximate the solution (u,p) by $(u^\varepsilon, p^\varepsilon)$ such that

$$-\Delta u^\varepsilon + \nabla p^\varepsilon = f \quad \text{in} \quad \Omega, \tag{6}$$

$$\operatorname{div} u^\varepsilon + \varepsilon p^\varepsilon = 0 \quad \text{in} \quad \Omega, \tag{7}$$

$$u^\varepsilon = 0 \quad \text{on} \quad \partial\Omega. \tag{8}$$

The weak form of (6)–(8) is to find $(u^\varepsilon, p^\varepsilon) \in X \times M$ such that

$$a(u^\varepsilon, v) - b(v, p^\varepsilon) = (f,v), \quad \forall v \in X, \tag{9}$$

$$b(u^\varepsilon, q) + \varepsilon(p^\varepsilon, q) = 0, \quad \forall q \in M. \tag{10}$$

Then, there holds the following error estimate (see [7]):

$$\|u - u^\varepsilon\|_0 + \|u - u^\varepsilon\|_1 + \|p - p^\varepsilon\|_0 \leqslant C\varepsilon. \tag{11}$$

Let T_h be a triangular partition of the domain $\Omega = \bigcup_{K \in T_h} K$ with the mesh size h, and

$$X_h = \Big\{v = (v^1, v^2); v\,|_K \in (P_1)^2, \int_F [v]\mathrm{d}s = 0, F \subset \partial K\Big\},$$

$$M_h = \Big\{q \in L^2(\Omega); q\,|_K \in P_0, \int_K q\mathrm{d}x\mathrm{d}y = 0\Big\}$$

be the approximation spaces for velocity and pressure, respectively, where $[v]$ denotes the jump value of v across the boundary F, and $[v] = v$ if $F \subset \partial\Omega$. $\Pi_h : (H_0^1(\Omega))^2 \to X_h$ and $J_h : L_0^2(\Omega) \to M_h$ are the associated interpolation operators. Then, for $(u,p) \in (H^2(\Omega))^2 \times H^1(\Omega)$, the following estimates hold[22]:

$$\|u - \Pi_h u\|_0 + h\|u - \Pi_h u\|_h \leqslant Ch^2 \|u\|_2, \tag{12}$$

$$\|p - J_h p\|_0 \leqslant Ch\|p\|_1, \tag{13}$$

where

$$\|v_h\|_h = \Big(\sum_{K\in T_h} |v_h|^2_{1,K} \Big)^{\frac{1}{2}}$$

for $v_h = (v_h^1, v_h^2)$.

The mixed finite element approximation of (9)–(10) is to find $(u_h^\varepsilon, p_h^\varepsilon) \in X_h \times M_h$ such that

$$a_h(u_h^\varepsilon, v_h) - b_h(v_h, p_h^\varepsilon) = (f, v_h), \quad \forall v_h \in X_h, \tag{14}$$
$$b_h(u_h^\varepsilon, q_h) + \varepsilon(p_h^\varepsilon, q_h) = 0, \quad \forall q_h \in M_h, \tag{15}$$

where

$$a_h(u_h, v_h) = \sum_{K\in T_h} \int_K \nabla u_h \nabla v_h \mathrm{d}x\mathrm{d}y,$$

$$b_h(v_h, q_h) = \sum_{K\in T_h} \int_K q_h \mathrm{div}\, v_h \mathrm{d}x\mathrm{d}y.$$

The following conclusions were derived in [9].

Lemma 2.1 *Assume that* (u,p) *and* $(u_h^\varepsilon,\ p_h^\varepsilon)$ *are the solutions of problems* (4)–(5) *and* (14)–(15), *respectively. There holds*

$$\|u - u_h^\varepsilon\|_h + \|p - p_h^\varepsilon\|_0 \leqslant C(h+\varepsilon)(\|u\|_2 + \|p\|_1). \tag{16}$$

The modified penalty method is used to approximate the solution (u,p) of (4)–(5) by the following pair (u_{mn}^h, p_{mn}^h):

$$u_{mn}^h = u_h^{\varepsilon_n} - \varepsilon_n \frac{u_h^{\varepsilon_m} - u_h^{\varepsilon_n}}{\varepsilon_m - \varepsilon_n}, \tag{17}$$

$$p_{mn}^h = p_h^{\varepsilon_n} - \varepsilon_n \frac{p_h^{\varepsilon_m} - p_h^{\varepsilon_n}}{\varepsilon_m - \varepsilon_n}, \tag{18}$$

where $(u_h^{\varepsilon_m}, p_h^{\varepsilon_m})$ and $(u_h^{\varepsilon_n}, p_h^{\varepsilon_n})$ are the solutions of (14)–(15) with $\varepsilon = \varepsilon_m$ and $\varepsilon = \varepsilon_n$, respectively. It is easy to show that (u_{mn}^h, p_{mn}^h) is the solution of the following equations:

$$a_h(u_{mn}^h, v_h) - b_h(v_h, p_{mn}^h) = (f, v_h), \quad \forall v_h \in X_h, \tag{19}$$

$$b_h(u_{mn}^h, q_h) = \frac{\varepsilon_m \varepsilon_n}{\varepsilon_m - \varepsilon_n}(p_h^{\varepsilon_m} - p_h^{\varepsilon_n}, q_h), \quad \forall q_h \in M_h. \tag{20}$$

Lemma 2.2 *When* $\varepsilon_m = k\varepsilon_n\ (k>1)$, *there holds*

$$\|u - u_{mn}^h\|_h + \|p - p_{mn}^h\|_0 \leqslant C(h + \varepsilon_m\varepsilon_n)(\|u\|_2 + \|p\|_1). \tag{21}$$

From (21), we can see that, if $\varepsilon_n = O(h^{\frac{1}{2}})$, the convergence order is $O(h)$, which is half order higher than that of the classical penalty method (14)–(15). In other words, the modified penalty method (17)–(18) with a larger penalty parameter can achieve the same accuracy as the classical method with a smaller penalty parameter.

3 Superconvergence analysis by L^2 projection method

In this section, we derive the superconvergence results of the nonconforming finite element penalty scheme for the Stokes problem by the L^2 projection method. The main idea is to project the finite element solution to another finite element space with a different coarse mesh. The difference in the mesh sizes can be used to achieve a better convergence rate after the post-processing procedure. Interested readers can refer to [16–19] and [21] for more details.

Let Γ_τ be another finite element partition with the mesh size τ, where $h \ll \tau$. Assume that τ and h have the following relation:

$$\tau = h^\alpha$$

with $\alpha \in (0,1)$. α plays an important role later in achieving the superconvergence.

Denote X_τ and M_τ by two finite element spaces consisting of piecewise polynomials of degrees r and t, respectively. Let Q_τ and R_τ be two L^2 projectors from $L^2(\Omega)$ onto the finite element spaces X_τ and M_τ, respectively, i.e.,

$$(Q_\tau u, v_h) = (u, v_h), \quad \forall v_h \in X_\tau$$

and

$$(R_\tau p, q_h) = (p, q_h), \quad \forall q_h \in M_\tau.$$

In what follows, we give the estimates for $u - Q_\tau u^h_{mn}$ and $p - R_\tau p^h_{mn}$, respectively.

Lemma 3.1 *Assume that* (u, p) *and* (u^h_{mn}, p^h_{mn}) *are determined by* (4)–(5) *and* (17)–(18), *respectively, and* $\varepsilon_m = k\varepsilon_n\,(k > 1)$. *Then, we have*

$$\|Q_\tau u - Q_\tau u^h_{mn}\|_0 \leqslant C(h^2 + h\varepsilon_m\varepsilon_n + h\varepsilon_n + \varepsilon_m\varepsilon_n)(\|u\|_2 + \|p\|_1), \tag{22}$$

$$|Q_\tau u - Q_\tau u^h_{mn}|_1 \leqslant C(h^{2-\alpha} + h^{1-\alpha}\varepsilon_m\varepsilon_n + h^{1-\alpha}\varepsilon_n + h^{-\alpha}\varepsilon_m\varepsilon_n)(\|u\|_2 + \|p\|_1). \tag{23}$$

Proof From the definition of Q_τ, we get

$$\begin{aligned}
&\|Q_\tau u - Q_\tau u^h_{mn}\|_0 \\
&= \sup_{\phi \in (L^2(\Omega))^2, \|\phi\|_0 \neq 0} \frac{|(Q_\tau u - Q_\tau u^h_{mn}, \phi)|}{\|\phi\|_0} \\
&= \sup_{\phi \in (L^2(\Omega))^2, \|\phi\|_0 \neq 0} \frac{|(u - u^h_{mn}, Q_\tau \phi)|}{\|\phi\|_0}.
\end{aligned} \tag{24}$$

Let (w, λ) be the solution of the following auxiliary problem:

$$-\Delta w + \nabla\lambda = Q_\tau\phi \quad \text{in} \quad \Omega, \tag{25}$$

$$\operatorname{div} w = 0 \quad \text{in} \quad \Omega, \tag{26}$$

$$w = 0 \quad \text{on} \quad \partial\Omega. \tag{27}$$

Then, there holds

$$\|w\|_2 + \|\lambda\|_1 \leqslant C\|Q_\tau\phi\|_0. \tag{28}$$

By (25)–(27), we have

$$a(w, v) - b(v, \lambda) = (Q_\tau\phi, v), \quad \forall v \in X, \tag{29}$$

$$b(w, q) = 0, \quad \forall q \in M. \tag{30}$$

Multiplying (25) by u_{mn}^h and (26) by p_{mn}^h, respectively, and integrating them over Ω give

$$\begin{aligned}&(Q_\tau\phi, u_{mn}^h)\\=&\,a_h(w,u_{mn}^h)-b_h(u_{mn}^h,\lambda)\\&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial w}{\partial n}u_{mn}^h\mathrm{d}s+\sum_{K\in T_h}\int_{\partial K}\lambda u_{mn}^h n\mathrm{d}s\end{aligned}\tag{31}$$

and

$$b_h(w,p_{mn}^h)=0.\tag{32}$$

Replacing v in (29) by u and q in (30) by p, and using (31)–(32), we obtain

$$\begin{aligned}&(Q_\tau\phi, u-u_{mn}^h)\\=&\,a_h(w,u-u_{mn}^h)-b_h(u-u_{mn}^h,\lambda)\\&+\sum_{K\in T_h}\int_{\partial K}\frac{\partial w}{\partial n}u_{mn}^h\mathrm{d}s-\sum_{K\in T_h}\int_{\partial K}\lambda u_{mn}^h n\mathrm{d}s\end{aligned}\tag{33}$$

and

$$b_h(w,p-p_{mn}^h)=0.\tag{34}$$

Similarly, it follows from (1)–(3), (19), and (20) that

$$\begin{aligned}&a_h(u-u_{mn}^h,v_h)-b_h(v_h,p-p_{mn}^h)\\&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}v_h\mathrm{d}s+\sum_{K\in T_h}\int_{\partial K}pv_h n\mathrm{d}s=0,\end{aligned}\tag{35}$$

and

$$b_h(u-u_{mn}^h,q_h)+\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p_h^{\varepsilon_m}-p_h^{\varepsilon_n},q_h)=0.\tag{36}$$

By use of (33)–(36), we have

$$\begin{aligned}&(Q_\tau\phi,u-u_{mn}^h)\\=&\,a_h(w-\Pi_h w,u-u_{mn}^h)+b_h(p-p_{mn}^h,\Pi_h w)+\sum_{K\in T_h}\int_{\partial K}\Big(\frac{\partial u}{\partial n}\Pi_h w-p\Pi_h wn\Big)\mathrm{d}s\\&-b_h(u-u_{mn}^h,\lambda-J_h\lambda)+\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p_h^{\varepsilon_m}-p_h^{\varepsilon_n},J_h\lambda)+\sum_{K\in T_h}\int_{\partial K}\Big(\frac{\partial w}{\partial n}u_{mn}^h-\lambda u_{mn}^h n\Big)\mathrm{d}s\\=&\,a_h(w-\Pi_h w,u-u_{mn}^h)-b_h(p-p_{mn}^h,w-\Pi_h w)+\sum_{K\in T_h}\int_{\partial K}\Big(\frac{\partial u}{\partial n}\Pi_h w-p\Pi_h wn\Big)\mathrm{d}s\\&-b_h(u-u_{mn}^h,\lambda-J_h\lambda)+\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p_h^{\varepsilon_m}-p_h^{\varepsilon_n},J_h\lambda)+\sum_{K\in T_h}\int_{\partial K}\Big(\frac{\partial w}{\partial n}u_{mn}^h-\lambda u_{mn}^h n\Big)\mathrm{d}s.\end{aligned}\tag{37}$$

For all $H\in L^2(F)$ and $F\subset\partial K$, let

$$P_0^F H=\frac{1}{|F|}\int_F H\mathrm{d}s.$$

It follows from the properties of X_h and M_h and the trace theorem that

$$\begin{aligned}&\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}\Pi_h w\mathrm{d}s\\&=\sum_{K\in T_h}\sum_{F\subset\partial K}\int_F\Big(\frac{\partial u}{\partial n}-P_0^F\frac{\partial u}{\partial n}\Big)(\Pi_h w-w)\mathrm{d}s\leqslant Ch^2\|u\|_2\|w\|_2.\end{aligned}\tag{38}$$

Similarly,

$$\sum_{K\in T_h}\int_{\partial K}p\Pi_h wn\mathrm{d}s\leqslant Ch^2\|p\|_1\|w\|_2.\tag{39}$$

Moreover, with the aid of (21), we have

$$\begin{aligned}\sum_{K\in T_h}\int_{\partial K}\frac{\partial w}{\partial n}u_{mn}^h\mathrm{d}s&=\sum_{K\in T_h}\sum_{F\subset\partial K}\int_F\Big(\frac{\partial w}{\partial n}-P_0^F\frac{\partial w}{\partial n}\Big)(u_{mn}^h-u)\mathrm{d}s\\&\leqslant Ch(h+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)\|w\|_2,\end{aligned}\tag{40}$$

and

$$\sum_{K\in T_h}\int_{\partial K}\lambda u_{mn}^h n\mathrm{d}s\leqslant Ch(h+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)\|\lambda\|_1.\tag{41}$$

At the same time, the application of (21) yields

$$\begin{aligned}&\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p_h^{\varepsilon_m}-p_h^{\varepsilon_n},J_h\lambda)\\&=\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p_h^{\varepsilon_m}-p,J_h\lambda)+\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p-p_h^{\varepsilon_n},J_h\lambda)\\&\leqslant\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(\|p_h^{\varepsilon_m}-p\|_0+\|p-p_h^{\varepsilon_n}\|_0)\|\lambda\|_0\\&\leqslant C(h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)\|\lambda\|_1.\end{aligned}\tag{42}$$

Using the Schwarz inequality, (12)–(13), (21), and (38)–(42), we have

$$\begin{aligned}&|(Q_\tau\phi,u-u_{mn}^h)|\\&\leqslant\|w-\Pi_h w\|_h\|u-u_{mn}^h\|_h+\|p-p_{mn}^h\|_0\|w-\Pi_h w\|_h+\|u-u_{mn}^h\|_h\|\lambda-J_h\lambda\|_0\\&\quad+C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)(\|w\|_2+\|\lambda\|_1)\\&\leqslant C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)(\|w\|_2+\|\lambda\|_1).\end{aligned}\tag{43}$$

From (28), we arrive at

$$\begin{aligned}&|(Q_\tau\phi,u-u_{mn}^h)|\\&\leqslant C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)\|Q_\tau\phi\|_0\\&\leqslant C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)\|\phi\|_0,\end{aligned}\tag{44}$$

which combining with (24) yields the desired result (22).

Furthermore, the inverse inequality gives

$$\begin{aligned}
&|Q_\tau u - Q_\tau u_{mn}^h|_1 \\
&\leqslant C\tau^{-1}\|Q_\tau u - Q_\tau u_{mn}^h\|_0 \\
&\leqslant C(h^2 + h\varepsilon_m\varepsilon_n + h\varepsilon_n + \varepsilon_m\varepsilon_n)\tau^{-1}(\|u\|_2 + \|p\|_1),
\end{aligned} \tag{45}$$

which along with $\tau = h^\alpha$ follows (23). The proof is completed.

Theorem 3.1 *Under the assumptions in Lemma* 3.1, *when the exact solution u is sufficiently smooth and $\varepsilon_n = O(h)$, there holds*

$$|u - Q_\tau u_{mn}^h|_1 \approx O(h^2) \quad as \quad r \longrightarrow \infty. \tag{46}$$

Proof Using the definition of Q_τ and Lemma 3.1, we have

$$\begin{aligned}
&|u - Q_\tau u_{mn}^h|_1 \\
&\leqslant |u - Q_\tau u|_1 + |Q_\tau u - Q_\tau u_{mn}^h|_1 \\
&\leqslant C\tau^r\|u\|_{r+1} + C(h^2 + h\varepsilon_m\varepsilon_n + h\varepsilon_n + \varepsilon_m\varepsilon_n)\tau^{-1}(\|u\|_2 + \|p\|_1) \\
&\leqslant Ch^{\alpha r}\|u\|_{r+1} + C(h^{2-\alpha} + h^{1-\alpha}\varepsilon_m\varepsilon_n + h^{1-\alpha}\varepsilon_n + h^{-\alpha}\varepsilon_m\varepsilon_n)(\|u\|_2 + \|p\|_1).
\end{aligned} \tag{47}$$

In fact, (47) is a superconvergence result for the velocity approximation.

For example, when $\varepsilon_n = O(h)$, with $r = 2$ and $\alpha = \frac{2}{3}$, we get

$$|u - Q_\tau u_{mn}^h|_1 \leqslant ch^{\frac{4}{3}}(\|u\|_3 + \|p\|_1).$$

With $r = 3$ and $\alpha = \frac{1}{2}$,

$$|u - Q_\tau u_{mn}^h|_1 \leqslant ch^{\frac{3}{2}}(\|u\|_4 + \|p\|_1).$$

With $r = 4$ and $\alpha = \frac{2}{5}$,

$$|u - Q_\tau u_{mn}^h|_1 \leqslant ch^{\frac{8}{5}}(\|u\|_5 + \|p\|_1).$$

Thus, when u is sufficiently smooth, it is easy to see that

$$|u - Q_\tau u_{mn}^h|_1 \approx O(h^2) \quad \text{as} \quad r \longrightarrow \infty.$$

Lemma 3.2 *Assume that (u, p) and (u_{mn}^h, p_{mn}^h) are determined by* (4)−(5) *and* (17)−(18), *respectively, and $\varepsilon_m = k\varepsilon_n$ $(k > 1)$. Then, we have*

$$\begin{aligned}
&\|R_\tau p - R_\tau p_{mn}^h\|_0 \\
&\leqslant C(h^{2-\alpha} + h^{1-\alpha}\varepsilon_m\varepsilon_n + h^{1-\alpha}\varepsilon_n + h^{-\alpha}\varepsilon_m\varepsilon_n)(\|u\|_2 + \|p\|_1).
\end{aligned} \tag{48}$$

Proof First, using the definition of R_τ, we get

$$\begin{aligned}
&\|R_\tau p - R_\tau p_{mn}^h\|_0 \\
&= \sup_{\phi \in L^2(\Omega), \|\phi\|_0 \neq 0} \frac{|(R_\tau p - R_\tau p_{mn}^h, \phi)|}{\|\phi\|_0} \\
&= \sup_{\phi \in L^2(\Omega), \|\phi\|_0 \neq 0} \frac{|(p - p_{mn}^h, R_\tau \phi)|}{\|\phi\|_0}.
\end{aligned} \tag{49}$$

Let (ϖ,χ) be the solution of the following auxiliary problem that is different from (25)–(27):

$$-\Delta\varpi+\nabla\chi=0 \quad \text{in} \quad \Omega, \tag{50}$$

$$\operatorname{div}\varpi=R_\tau\phi \quad \text{in} \quad \Omega, \tag{51}$$

$$\varpi=0 \quad \text{on} \quad \partial\Omega. \tag{52}$$

Then, there holds[16]

$$\|\varpi\|_2+\|\chi\|_1\leqslant C\|R_\tau\phi\|_1. \tag{53}$$

By (50)–(52), we get

$$a(\varpi,v)-b(v,\chi)=0,\quad \forall v\in X, \tag{54}$$

$$b(\varpi,q)=(R_\tau\phi,q),\quad \forall q\in M. \tag{55}$$

Multiplying (50) by u^h_{mn} and (51) by p^h_{mn} and integrating them over Ω give

$$\begin{aligned}&a_h(\varpi,u^h_{mn})-b_h(u^h_{mn},\chi)\\&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial\varpi}{\partial n}u^h_{mn}\mathrm{d}s+\sum_{K\in T_h}\int_{\partial K}\chi u^h_{mn}n\mathrm{d}s=0\end{aligned} \tag{56}$$

and

$$b_h(\varpi,p^h_{mn})=(R_\tau\phi,p^h_{mn}). \tag{57}$$

Taking $(v,q)=(u,p)$ in (54)–(55) and using (56)–(57), we obtain

$$\begin{aligned}&a_h(\varpi,u-u^h_{mn})-b_h(u-u^h_{mn},\chi)\\&+\sum_{K\in T_h}\int_{\partial K}\frac{\partial\varpi}{\partial n}u^h_{mn}\mathrm{d}s-\sum_{K\in T_h}\int_{\partial K}\chi u^h_{mn}n\mathrm{d}s=0\end{aligned} \tag{58}$$

and

$$b_h(\varpi,p-p^h_{mn})=(R_\tau\phi,p-p^h_{mn}). \tag{59}$$

From (35)–(36) and (58)–(59), we have

$$\begin{aligned}&(R_\tau\phi,p-p^h_{mn})\\=&\,b_h(\varpi-\Pi_h\varpi,p-p^h_{mn})+a_h(\Pi_h\varpi,u-u^h_{mn})\\&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}\Pi_h\varpi\mathrm{d}s+\sum_{K\in T_h}\int_{\partial K}p\Pi_h\varpi n\mathrm{d}s\\=&\,b_h(\varpi-\Pi_h\varpi,p-p^h_{mn})+a_h(\Pi_h\varpi-\varpi,u-u^h_{mn})+a_h(\varpi,u-u^h_{mn})\\&-\sum_{K\in T_h}\int_{\partial K}\frac{\partial u}{\partial n}\Pi_h\varpi\mathrm{d}s+\sum_{K\in T_h}\int_{\partial K}p\Pi_h\varpi n\mathrm{d}s\\=&\,b_h(\varpi-\Pi_h\varpi,p-p^h_{mn})+a_h(\Pi_h\varpi-\varpi,u-u^h_{mn})+b_h(u-u^h_{mn},\chi)\\&-\sum_{K\in T_h}\int_{\partial K}\Big(\frac{\partial\varpi}{\partial n}u^h_{mn}-\chi u^h_{mn}n+\frac{\partial u}{\partial n}\Pi_h\varpi-p\Pi_h\varpi n\Big)\mathrm{d}s\\=&\,b_h(\varpi-\Pi_h\varpi,p-p^h_{mn})+a_h(\Pi_h\varpi-\varpi,u-u^h_{mn})\\&+b_h(u-u^h_{mn},\chi-J_h\chi)-\frac{\varepsilon_m\varepsilon_n}{\varepsilon_m-\varepsilon_n}(p^{\varepsilon_m}_h-p^{\varepsilon_n}_h,J_h\chi)\\&-\sum_{K\in T_h}\int_{\partial K}\Big(\frac{\partial\varpi}{\partial n}u^h_{mn}-\chi u^h_{mn}n+\frac{\partial u}{\partial n}\Pi_h\varpi-p\Pi_h\varpi n\Big)\mathrm{d}s.\end{aligned} \tag{60}$$

With the similar arguments of (38)–(42), the Schwarz inequality, the interpolation theorem, and (21), we have

$$
\begin{aligned}
&|(R_\tau\phi, p-p_{mn}^h)|\\
\leqslant&\|\varpi-\Pi_h\varpi\|_h\|p-p_{mn}^h\|_0+\|u-u_{mn}^h\|_h\|\varpi-\Pi_h\varpi\|_h+\|u-u_{mn}^h\|_h\|\chi-J_h\chi\|_0\\
&+C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)(\|\varpi\|_2+\|\chi\|_1)\\
\leqslant&C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)(\|\varpi\|_2+\|\chi\|_1).
\end{aligned}\tag{61}
$$

Moreover, (53) and the inverse inequality give

$$
\begin{aligned}
&|(R_\tau\phi, p-p_{mn}^h)|\\
\leqslant&\, C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)(\|u\|_2+\|p\|_1)\|R_\tau\phi\|_1\\
\leqslant&\, C(h^2+h\varepsilon_m\varepsilon_n+h\varepsilon_n+\varepsilon_m\varepsilon_n)\tau^{-1}(\|u\|_2+\|p\|_1)\|\phi\|_0,
\end{aligned}\tag{62}
$$

which combining with (49) and $\tau=h^\alpha$ yields (48). The proof is completed.

Theorem 3.2 *Under the assumptions of Lemma* 3.2, *when the exact solution p is sufficiently smooth and $\varepsilon_n=O(h)$, there holds*

$$
\|p-R_\tau p_{mn}^h\|_0\approx O(h^2)\quad as\quad t\longrightarrow\infty.\tag{63}
$$

Proof The definition of R_τ and Lemma 3.2 give

$$
\begin{aligned}
&\|p-R_\tau p_{mn}^h\|_0\\
\leqslant&\,\|p-R_\tau p\|_0+\|R_\tau p-R_\tau p_{mn}^h\|_0\\
\leqslant&\, Ch^{\alpha(t+1)}\|p\|_{t+1}+C(h^{2-\alpha}+h^{1-\alpha}\varepsilon_m\varepsilon_n+h^{1-\alpha}\varepsilon_n+h^{-\alpha}\varepsilon_m\varepsilon_n)\\
&\cdot(\|u\|_2+\|p\|_1),
\end{aligned}\tag{64}
$$

which shows an improvement of order for the pressure approximation.

For example, when $\varepsilon_n=O(h)$, with $t=1$ and $\alpha=\frac{2}{3}$, we have

$$
\|p-R_\tau p_{mn}^h\|_0\leqslant ch^{\frac{4}{3}}(\|u\|_2+\|p\|_2).
$$

With $t=2$ and $\alpha=\frac{1}{2}$, we have

$$
\|p-R_\tau p_{mn}^h\|_0\leqslant ch^{\frac{3}{2}}(\|u\|_2+\|p\|_3).
$$

With $t=3$ and $\alpha=\frac{2}{5}$, we have

$$
\|p-R_\tau p_{mn}^h\|_0\leqslant ch^{\frac{8}{5}}(\|u\|_2+\|p\|_4).
$$

With $t=4$ and $\alpha=\frac{1}{3}$, we have

$$
\|p-R_\tau p_{mn}^h\|_0\leqslant ch^{\frac{5}{3}}(\|u\|_2+\|p\|_5).
$$

Thus, when the exact solution p is sufficiently smooth, it is easy to obtain that

$$
\|p-R_\tau p_{mn}^h\|_0\approx O(h^2)\quad \text{as}\quad t\longrightarrow\infty.
$$

Remark 3.1 Theorems 3.1–3.2 reveal that the $O(h^k)\,(1<k<2)$-order superconvergence for velocity and pressure can be obtained for the modified penalty scheme by the L^2 projection method, in which the penalty parameter only needs to be chosen to be of order $O(h)$. In fact, using the L^2 projection method and referring to the analysis in [6] for the conforming P_1b-P_1[23] finite element space, we can also obtain the $O(h^k)\,(1 < k < 2)$-order superconvergence for the classical penalty scheme with the nonconforming finite element space discussed above. However, the penalty parameter must be chosen to be of order $O(h^k)\,(1 < k < 2)$, which is smaller than that of the modified penalty scheme.

Remark 3.2 From the above analysis of Lemmas 3.1–3.2 and Theorems 3.1–3.2, one can easily check that, if the nonconforming finite element space X_h satisfies the following conditions:

(i) the polynomial space in the construction of X_h contains P_1,

(ii) $\int_F [v_h]\mathrm{d}s = 0, \forall v_h \in X_h$, and $F \subset \partial K$,

(iii) the spaces X_h and M_h satisfy the B-B condition, i.e., there exists a constant $\beta > 0$ such that $\sup\limits_{v_h\in X_h} \dfrac{b(v_h, q_h)}{\|v\|_h} \geqslant \beta\|q_h\|_0, \forall q_h \in M_h$,

then the superconvergence results in this paper are also valid, such as the Q_1^{rot} element space[24], the EQ_1^{rot} element space[25–26], and the extended Crouzeix-Raviart element space[27]. Thus, the proposed method is applicable in practical applications.

4 Numerical experiment

In this section, we present some numerical results to confirm the present theoretical analysis.

Let $u = (u^1, u^2)$. Then, we adopt the example with the exact solution (u, p) of problem (1)–(3):

$$\begin{aligned}
&u^1 = 100(x^4 - 2x^3 + x^2)(4y^3 - 6y^2 + 2y),\\
&u^2 = -100(y^4 - 2y^3 + y^2)(4x^3 - 6x^2 + 2x),\\
&p = x^5 + y^5 - \frac{1}{3},
\end{aligned}$$

while $f = (f^1, f^2)$ can be stated as

$$\begin{aligned}
f^1 =& -100(12x^2 - 12x + 2)(4y^3 - 6y^2 + 2y)\\
&- 100(x^4 - 2x^3 + x^2)(24y - 12) + 5x^4,\\
f^2 =& \,100(24x - 12)(y^4 - 2y^3 + y^2)\\
&+ 100(4x^3 - 6x^2 + 2x)(12y^2 - 12y + 2) + 5y^4.
\end{aligned}$$

We divide the domain Ω into a family of quasi-uniform triangles with N nodes. As for the classical penalty method (14)–(15), we obtain the errors of velocity u and pressure p under the the energy norm and the L^2 norm with the penalty parameter $\epsilon = 1.0\mathrm{E}-4$ in Table 1. It is clear that the energy norm for velocity and the L^2 norm for pressure can achieve the optimal convergence order with $O(h)$.

Table 1 Errors of velocity u and pressure p with $\epsilon = 1.0\mathrm{E}-4$

Number of nodes	$\|u - u_h^\varepsilon\|_h$	Order	$\|p - p_h^\varepsilon\|_0$	Order
9	5.9161		1.5377	
25	3.2982	0.8429	8.4931E–1	0.8564
81	1.7294	0.9315	4.4346E–1	0.9375
289	0.8778	0.9783	2.1378E–1	1.0527
1089	4.4075E–1	0.9939	1.0451E–1	1.0324
4225	2.2062E–1	0.9984	5.1856E–2	1.0111
16641	1.1034E–1	0.9996	2.5870E–2	1.0032

For the modified penalty method (17)–(18), we present the errors of velocity u and pressure p under the energy norm and the L^2 norm with the penalty parameters $\epsilon_n = 1.0\mathrm{E}-2$ and $\epsilon_m = 5.0\mathrm{E}-2$ in Table 2. The numerical results indicate that the values of $\epsilon_n = 1.0\mathrm{E}-2$ and $\epsilon_m = 5.0\mathrm{E}-2$ are sufficient for the modified penalty method. However, to obtain the same convergence order for the classical penalty method, we need to choose the penalty parameter $\epsilon = 1.0$E–4 (see Table 1). From this point of view, we can see that the modified penalty method is more efficient than the classical penalty method.

Table 2 Errors of velocity u and pressure p based on u^h_{mn} and p^h_{mn} with $\epsilon_n = 1.0$E–2 and $\epsilon_m = 5.0$E–2

Number of nodes	$\|u-u^h_{mn}\|_h$	Order	$\|p-p^h_{mn}\|_0$	Order
9	5.916 1		1.537 5	
25	3.298 2	0.842 9	8.491 5E–1	0.856 4
81	1.729 4	0.931 5	4.434 2E–1	0.937 3
289	0.877 8	0.978 3	2.138 1E–1	1.052 4
1 089	4.407 5E–1	0.993 9	1.045 6E–1	1.031 9
4 225	2.206 2E–1	0.998 4	5.191 0E–2	1.010 3
16 641	1.103 4E–1	0.999 6	2.5937 E–2	1.001 0

In Table 3, we present the numerical results for the L^2 projection to the higher-order finite element space on a coarse mesh. We take $\epsilon_n = 1.0\mathrm{E}-3$ and $\epsilon_m = 5.0\mathrm{E}-3$ with $\tau = h^{\frac{1}{2}}$, $r = 3$, and $t = 2$ for example. It is clear that the superconvergence results of the energy norm for velocity and the L^2 norm for pressure are derived. This demonstrates the validity of the present theoretical analysis in Theorems 3.1–3.2.

Table 3 Superconvergence results of velocity u and pressure p by L^2 projection with $\epsilon_n = 1.0\mathrm{E}-3$ and $\epsilon_m = 5.0\mathrm{E}-3$

Number of nodes	$\|u-Q_\tau u^h_{mn}\|_1$	Order	$\|p-R_\tau p^h_{mn}\|_0$	Order
25	1.640 797 111 395		0.456 076 423 989	
289	0.156 576 173 253	1.694 7	0.071 906 266 858	1.332 5
4 225	0.020 577 742 912	1.463 9	0.005 787 531 056	1.817 5
66 049	0.002 552 947 306	1.505 4	0.000 444 328 743	1.851 6

Moreover, in Table 4, we also present the errors of velocity u and pressure p under the energy norm and the L^2 norm for the classical penalty method with the penalty parameter $\epsilon = O(h)$. We can see that the optimal convergence order with $O(h)$ is derived, which coincides with the theoretical analysis in Lemma 2.1.

Table 4 Errors of velocity u and pressure p with $\epsilon = O(h)$

Number of nodes	$\|u-u^\epsilon_h\|_h$	Order	$\|p-p^\epsilon_h\|_0$	Order
25	3.299 4		7.850 6E–1	
81	1.729 7	0.931 7	4.303 4E–1	0.867 3
289	8.779 4E–1	0.978 3	2.128 4E–1	1.015 7
1 089	4.408 3E–1	0.993 9	1.051 8E–1	1.016 9
4 225	2.206 6E–1	0.998 4	5.238 4E–2	1.005 6
16 641	1.103 6E–1	0.999 6	2.616 5E–2	1.001 5
66 049	5.518 5E–2	0.999 9	1.307 9E–2	1.000 4

In Table 5, we present the errors of velocity u and pressure p under the energy norm and the L^2 norm for the modified penalty method (17)–(18) with the penalty parameter $\epsilon_n = O(h^{\frac{1}{2}})$. The optimal convergence results with order $O(h)$ are obtained, in which the penalty parameter only needs to be chosen to be of order $O(h^{\frac{1}{2}})$ instead of $O(h)$ in the classical penalty method (see Table 4). The numerical results in Table 5 confirm the theoretical analysis in Lemma 2.2.

Table 5 Errors of velocity u and pressure p based on u_{mn}^h and p_{mn}^h with $\epsilon_n = O(h^{\frac{1}{2}})$

Number of nodes	$\|u - u_{mn}^h\|_h$	Order	$\|p - p_{mn}^h\|_0$	Order
25	3.298 6		8.171 4E–1	
81	1.729 5	0.931 4	4.383 4E–1	0.898 5
289	8.780 0E–1	0.978 1	2.162 2E–1	1.019 5
1 089	4.409 4E–1	0.993 6	1.077 8E–1	1.004 3
4 225	2.207 7E–1	0.998 0	5.446 9E–2	0.984 6
16 641	1.104 4E–1	0.999 1	2.767 1E–2	0.977 0
66 049	5.524 3E–2	0.999 4	1.407 1E–2	0.975 5

In Table 6, we present the superconvergence results for the L^2 projection to the higher-order finite element space on a coarse mesh. We set $\epsilon_n = O(h)$ with $\tau = h^{\frac{1}{2}}$, $r = 3$, and $t = 2$. We can see that the superconvergence results of the energy norm for velocity and the L^2 norm for pressure are derived, and the numerical results coincide with the theoretical analysis in Theorems 3.1–3.2.

Table 6 Superconvergence results of velocity u and pressure p by L^2 projection with $\epsilon_n = O(h)$

Number of nodes	$\|u - Q_\tau u_{mn}^h\|_1$	Order	$\|p - R_\tau p_{mn}^h\|_0$	Order
25	1.641 017 808 116		0.448 310 708 306	
289	0.156 604 394 388	1.694 6	0.072 508 664 869	1.314 1
4 225	0.020 583 418 224	1.463 7	0.005 895 378 142	1.810 2
66 049	0.002 553 048 157	1.505 5	0.000 447 443 774	1.859 9

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