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Propagation of *P*- and *S*-waves in initially stressed gravitating half space*

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Abstract The present paper contributes in studying the phase velocities of P- and S-waves in a half space subjected to a compressive initial stress and gravity field. The density and acceleration due to gravity vary quadratically along the depth. The dispersion equation is derived in a closed form. It is shown that the phase velocities depend not only on the initial stress, gravity, and direction of propagation but also on the inhomogeneity parameter associated with the density and acceleration due to gravity. Various particular cases are obtained, and the results match with the classical results. Numerical investigations on the phase velocities of P- and S-waves against the wave number are made for various sets of values of the material parameters, and the results are illustrated graphically. The graphical user interface model is developed to generalize the effect.

Key words P-wave, S-wave, initial stress, gravity, half space, graphical user interface

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1 Introduction

For engineers, physicists, and seismologists, the study of stress waves always is a great deal of interest. The theoretical studies are helpful in forecasting geophysical parameters at deep depths through signal processing and seismic data analysis. Metallurgists use this for the analysis of rock and material structures through non-destructive testing (NDT). The knowledge of seismic waves is helpful in investigating the exploration of oil, underground water, and gas accumulation. In recent years, efforts have been made in using seismic methods to characterize hydrocarbon reservoirs, to monitor reservoir production, and to enhance oil recovery processes. Our globe is a spherical body with finite dimension, and the generated elastic waves must receive the effect of the boundaries. Naturally, this concept leads us to the investigation of boundary waves or surface waves, which are confined to some surface during their propagation. In fact, the study of surface waves in homogenous, heterogeneous, and layered media has not been of central interest to theoretical seismologists until recently. The study of generation and propagation of waves in elastic solid has a long and distinguished history in the field of seismology and geophysics as well as in applied mathematics. The extensive literature on the subject is available in the books of Ewing et al.[1], Bullen[2], Bath[3], Miklowitz[4], Achenbach[5], Kalski[6], and many other authors.

Any disturbance in the earth interior may serve as the basic reason of seismic wave propagation. The presence of the gravity field and varying material properties has remarkable effects

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on the propagation of the elastic waves. The acceleration due to gravity g plays an important role in studying the dynamic and static problems of the earth. Anisotropy is a general property of geological media. Transverse isotropy, the simplest form of anisotropy which characterizes media with a single symmetry axis, can be used to describe anisotropy in many real media of geophysical interest. So far, studies of transversely isotropic media based on the analysis of P-waves have assumed that the travel time integrates the anisotropic properties along the geometrical rays. This approximation may possibly lead to important bases, particularly for very long paths which are associated to large Fresnel zones.

The earth is an initially stressed medium. Due to the variations of temperature, gravitational pull, slow process of creep, and the pressure due to over burden layer, a large quantity of stresses (may be called as initial stresses) are stored in the layers of the earth. For simplicity, if we consider the earth to be elastic, these initial stresses would modify the elastic coefficients of the earth and would have effects in the propagation of body and surface waves inside the earth. The present paper attempts at showing the effects of the initial stresses, gravitational pull (Biot's gravity parameter), various inhomogeneity parameter involved in g and the density ρ of the medium on the propagation of P- and S-waves. Further, the acceleration due to gravity inside the earth varies, and the variation is found to be quadratic. The density of the material of the earth is also found to vary quadratically. Attempts have also been made to show the effects of these variations on the propagation of body waves.

Many different methods proposed for modeling waves in heterogeneous media have their own ranges of validity and interest. Numerical integration over the wave number was used by Aki and Larner[7] and Bard and Bouchon[8], who introduced the Rayleigh ansatz for the diffraction sources, in order to model laterally heterogeneous media. Plane dynamic problems for elastic incompressible bodies with initial stresses were studied by Babich and Guz[9]. Alekseev and Mikhailenko[10] studied dynamic problems of elastic wave propagation in inhomogeneous media. Borcherdt[11] studied reflection and refraction of type II S-waves in elastic and inelastic media. Borcherdt[12] also studied reflection and refraction of general P- and type I S-waves in elastic and inelastic solid. Further, reflection and refraction for various elastic waves were dealt by Chattopadhyay et al.[13–14]. P- and S-waves in a medum under initial stresses and under gravity were studied by Dey and Dutta[15].

The problem deals with the propagation of P- and S-waves in a gravitating elastic medium in the presence of initial normal and shear stresses taking quadratic variations for the gravity and density. After getting the analytical solution, the numerical values of P- and S-waves are computed numerically, taking various values of different parameters, and presented by graphs. The velocities are observed to be much affected by the varying gravity and density. Damping is also observed.

2 Formulation of problem

Consider a gravitating half space under the initial compressive stress P_0 in the x-direction. The gravity field will further generate the hydrostatic compressive stress $-\rho g z$ in all directions, ρ and g are the density and the acceleration due to gravity at a distance z from the free surface of the earth, which are taken to have quadratic variations with respect to the distance from the free surface. Taking the x-axis on the surface and the z-axis perpendicular to the surface and directed upward with the origin O as shown in Fig. 1, the initial normal stress field of the system may be written as follows:

$$\begin{cases} S_{11} = -P_0 + \rho g z, \\ S_{22} = \rho g z, \\ S_{33} = \rho g z, \end{cases} \tag{1}$$

where we assume

$$\begin{cases} g = g_0(1 + az + b^2z^2), \\ \rho = \rho_0(1 - \alpha z - \beta^2 z^2), \end{cases} \tag{2}$$

where ρ_0 and g_0 are the density and the acceleration due to gravity at the free surface of the medium. Also, a, α, b, and β are constants having dimensions equal to those of the inverse of the length. In addition to the above normal stress field, the medium may subject to a constant shear stress (or the pre-stressed parameter) of the magnitude S_{12}. Let a ray of wave propagate in the medium originating from some source at a considerable distance. Let the direction of the wave make an angle θ with the z-axis.

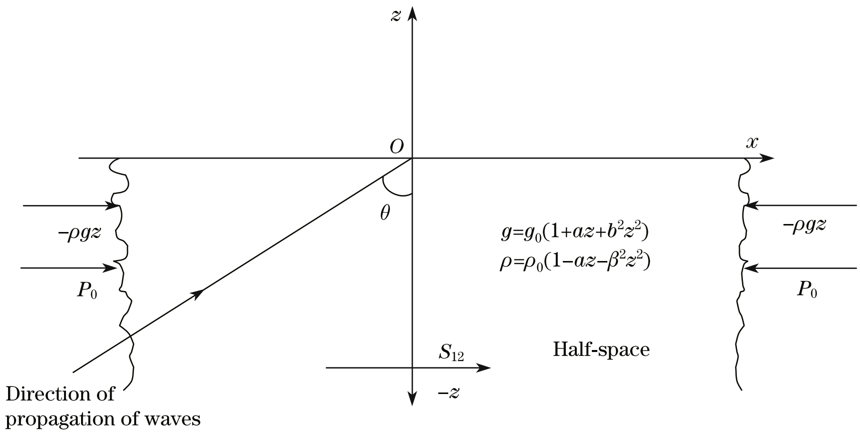

Fig. 1 Geometry of problem

3 Solution problem

The equation of motion under the above mentioned stress field may be written as follows[16]:

$$\begin{cases} \dfrac{\partial S_{11}}{\partial x} + \dfrac{\partial S_{12}}{\partial z} - \rho g(\omega + e_{xz}) - P_0 \dfrac{\partial \omega}{\partial z} - S_{12}\Big(\dfrac{\partial^2 u_2}{\partial x^2} - \dfrac{\partial^2 u_1}{\partial x \partial z}\Big) = \rho \dfrac{\partial^2 u_1}{\partial t^2}, \\ \dfrac{\partial S_{12}}{\partial x} + \dfrac{\partial S_{22}}{\partial z} - P_0 \dfrac{\partial \omega}{\partial x} + S_{12}\Big(\dfrac{\partial^2 u_2}{\partial x \partial z} - \dfrac{\partial^2 u_1}{\partial z^2}\Big) + \rho g \dfrac{\partial u_1}{\partial x} = \rho \dfrac{\partial^2 u_2}{\partial t^2}, \end{cases} \tag{3}$$

where s_{ij} are the incremental stresses, u_i are the displacement components, and

$$\omega = \frac{1}{2}\Big(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial z}\Big), \quad e_{xz} = \frac{1}{2}\Big(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial x}\Big).$$

The effective elastic constant coefficients in an initially stressed medium obtained by Dey and Mahto[17] are given for $\lambda = \mu$ as follows:

$$\begin{cases} B_{11} = 3\mu\Big(1 - \dfrac{\rho g z}{5\mu} - \dfrac{3}{5}\dfrac{P_0}{\mu}\Big), \quad B_{12} = \mu\Big(1 - \dfrac{6}{5}\dfrac{\rho g z}{\mu} - \dfrac{9}{10}\dfrac{P_0}{\mu}\Big), \\ B_{22} = 3\mu\Big(1 - \dfrac{\rho g z}{5\mu} + \dfrac{2}{5}\dfrac{P_0}{\mu}\Big), \quad Q = \mu\Big(1 + \dfrac{3}{10}\dfrac{\rho g z}{\mu} - \dfrac{7}{20}\dfrac{P_0}{\mu} + \dfrac{S_{12}}{2\mu}\Big). \end{cases} \tag{4}$$

Let us suppose

$$\overline{X} = (1 + az + b^2z^2)(1 - \alpha z - \beta^2 z^2)z, \tag{5}$$

and consider

$$G = \frac{\rho_0 g_0}{K\mu}, \quad \overline{Z} = -z, \quad \xi = \frac{P_0}{2\mu}, \quad \xi_3 = \frac{S_{12}}{2\mu},$$

where K is the wave number.

Now, the elastic coefficients transform to

$$\begin{cases} \dfrac{B_{11}}{\mu} = 3\Big(1 - \dfrac{GK\overline{X}}{5} - \dfrac{6}{5}\xi\Big), & \dfrac{B_{12}}{\mu} = 1 - \dfrac{6}{5}GK\overline{X} - \dfrac{9}{5}\xi, \\ \dfrac{B_{22}}{\mu} = 3\Big(1 - \dfrac{GK\overline{X}}{5} + \dfrac{4}{5}\xi\Big), & \dfrac{Q}{\mu} = 1 + \dfrac{3}{10}GK\overline{X} - \dfrac{7}{10}\xi + \xi_3. \end{cases} \tag{6}$$

Also, we have

$$\frac{\partial B_{11}}{\partial x} = \frac{\partial B_{12}}{\partial x} = \frac{\partial B_{22}}{\partial x} = \frac{\partial Q}{\partial x} = 0.$$

Write

$$\begin{cases} \overline{X} = (1 + (a-\alpha)z - (\beta^2 + \alpha a - b^2)z^2 - (a\beta^2 + \alpha b^2)z^3 - b^2\beta^2 z^4)z, \\ \overline{Y} = \dfrac{\partial \overline{X}}{\partial z} = 1 + 2(a-\alpha)z - 3(\beta^2 + \alpha a - b^2)z^2 - 4(a\beta^2 + \alpha b^2)z^3 - 5b^2\beta^2 z^4, \end{cases} \tag{7}$$

where $\overline{Y}$ is a dimensionless quantity.

Then, we have

$$\begin{cases} \dfrac{\partial B_{11}}{\partial z} = \dfrac{3}{5}GK\overline{Y}\mu, & \dfrac{\partial B_{12}}{\partial z} = \dfrac{6}{5}GK\overline{Y}\mu, \\ \dfrac{\partial B_{22}}{\partial z} = \dfrac{3}{5}GK\overline{Y}\mu, & \dfrac{\partial Q}{\partial z} = -\dfrac{3}{10}GK\overline{Y}\mu. \end{cases} \tag{8}$$

3.1 Stress displacement relations

The relation between the incremental stresses S_{ij} and the displacement components u_1 and u_2 may be written as

$$\begin{cases} S_{11} = B_{11}\dfrac{\partial u_1}{\partial x} + B_{12}\dfrac{\partial u_2}{\partial z}, \\ S_{12} = Q\Big(\dfrac{\partial u_1}{\partial z} + \dfrac{\partial u_2}{\partial x}\Big), \\ S_{22} = (B_{12} - P_0)\dfrac{\partial u_1}{\partial x} + B_{22}\dfrac{\partial u_2}{\partial z}. \end{cases} \tag{9}$$

Using Eq. (9) in Eq. (3), we get

$$\begin{aligned} &B_{11}\frac{\partial^2 u_1}{\partial x^2} + \Big(B_{12} + Q - \frac{P_0}{2}\Big)\frac{\partial^2 u_2}{\partial z \partial x} + \Big(Q + \frac{P_0}{2}\Big)\frac{\partial^2 u_1}{\partial z^2} - S_{12}\Big(\frac{\partial^2 u_2}{\partial x^2} - \frac{\partial^2 u_1}{\partial x \partial z}\Big) \\ &- \Big(\rho g - \frac{\partial Q}{\partial z}\Big)\frac{\partial u_2}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial u_1}{\partial z} = \rho\frac{\partial^2 u_1}{\partial t^2} \end{aligned} \tag{10}$$

and

$$\begin{aligned} &B_{22}\frac{\partial^2 u_2}{\partial z^2} + \Big(B_{12} + Q - \frac{P_0}{2}\Big)\frac{\partial^2 u_1}{\partial x \partial z} + \Big(Q - \frac{P_0}{2}\Big)\frac{\partial^2 u_2}{\partial x^2} + \Big(\frac{\partial B_{12}}{\partial z} + \rho g\Big)\frac{\partial u_1}{\partial x} \\ &+ \frac{\partial B_{22}}{\partial z}\frac{\partial u_2}{\partial z} + S_{12}\Big(\frac{\partial^2 u_2}{\partial x \partial z} - \frac{\partial^2 u_1}{\partial z^2}\Big) + \frac{\partial Q}{\partial x}\Big(\frac{\partial u_1}{\partial z} + \frac{\partial u_2}{\partial x}\Big) = \rho\frac{\partial^2 u_2}{\partial t^2}. \end{aligned} \tag{11}$$

For the plane wave solution, we take

$$u_1 = A\mathrm{e}^{\mathrm{i}K(x_1p_1+x_2p_2-ct)},$$

$$u_2 = B\mathrm{e}^{\mathrm{i}K(x_1p_1+x_2p_2-ct)},$$

where

$$p_1^2 + p_2^2 = 1.$$

i.e.,

$$p_1 = \cos\theta, \quad p_2 = \sin\theta,$$

and c is a velocity of propagation.

Substituting these into Eq. (10) and Eq. (11), we get

$$A\Big(B_{11}p_1^2 + \Big(Q + \frac{P_0}{2}\Big)p_2^2 + S_{12}p_1p_2 - \frac{\mathrm{i}}{K}\frac{\partial Q}{\partial z}p_2 - \rho c^2\Big)$$

$$+ B\Big(\Big(B_{12} + Q - \frac{P_0}{2}\Big)p_1p_2 - S_{12}p_1^2 + \frac{\mathrm{i}}{K}\Big(\rho g - \frac{\partial Q}{\partial z}\Big)p_1\Big) = 0$$

and

$$A\Big(\Big(B_{12} + Q - \frac{P_0}{2}\Big)p_1p_2 - S_{12}p_2^2 - \frac{\mathrm{i}}{K}\Big(p_1\Big(\frac{\partial B_{12}}{\partial z} + \rho g\Big) + \frac{\partial Q}{\partial x}p_2\Big)\Big)$$

$$+ B\Big(B_{22}p_2^2 + \Big(Q + \frac{P_0}{2}\Big)p_1^2 + S_{12}p_1p_2 - \frac{\mathrm{i}}{K}\Big(p_2\frac{\partial B_{22}}{\partial z} + p_1\frac{\partial Q}{\partial x}\Big) - \rho c^2\Big) = 0.$$

For non-zero solutions, eliminating A and B, we get the velocity equation as follows:

$$\begin{vmatrix} \begin{array}{l} B_{11}p_1^2 + \Big(Q + \frac{P_0}{2}\Big)p_2^2 + S_{12}p_1p_2 \\ -\frac{\mathrm{i}}{K}\frac{\partial Q}{\partial z}p_2 - \rho c^2 \end{array} & \begin{array}{l} \Big(B_{12} + Q - \frac{P_0}{2}\Big)p_1p_2 - S_{12}p_1^2 \\ +\frac{\mathrm{i}}{K}\Big(\rho g - \frac{\partial Q}{\partial z}\Big)p_1 \end{array} \\ \begin{array}{l} \Big(B_{12} + Q - \frac{P_0}{2}\Big)p_1p_2 - p_2^2S_{12} \\ -\frac{\mathrm{i}}{K}\Big(p_1\Big(\frac{\partial B_{12}}{\partial z} + \rho g\Big) + \frac{\partial Q}{\partial x}p_2\Big) \end{array} & \begin{array}{l} B_{22}p_2^2 + \Big(Q + \frac{P_0}{2}\Big)p_1^2 + S_{12}p_1p_2 \\ -\frac{\mathrm{i}}{K}\Big(p_2\frac{\partial B_{22}}{\partial z} + p_1\frac{\partial Q}{\partial x}\Big) - \rho c^2 \end{array} \end{vmatrix} = 0, \quad (12)$$

which on expansion and rearrangement takes the form

$$\rho^2c^4 - \rho(r_1 - \mathrm{i}r_2)c^2 + (S_1 + \mathrm{i}S_2) = 0, \tag{13}$$

where

$$r_1 = \Big(\Big(B_{11} + Q + \frac{P_0}{2}\Big)p_1^2 + \Big(B_{22} + Q + \frac{P_0}{2}\Big)p_2^2 + 2S_{12}p_1p_2\Big),$$

$$
\begin{aligned}
r_2 =& \frac{1}{K}\Big(p_2\frac{\partial B_{22}}{\partial z}+p_1\frac{\partial Q}{\partial x}+p_2\frac{\partial Q}{\partial z}\Big)\\
=& \frac{1}{K}\Big(\frac{3}{5}GK\overline{Y}\mu-\frac{3}{10}GK\overline{Y}\mu\Big)p_2\\
=& \frac{3}{10}G\overline{Y}\mu p_2,
\end{aligned}
$$

$$
\begin{aligned}
S_1 =& \Big(B_{11}p_1^2+\Big(Q+\frac{P_0}{2}\Big)p_2^2+S_{12}p_1p_2\Big)\Big(B_{22}p_2^2+\Big(Q+\frac{P_0}{2}\Big)p_1^2+S_{12}p_1p_2\Big)\\
&-\frac{1}{K^2}p_2\frac{\partial Q}{\partial z}\Big(p_2\frac{\partial B_{22}}{\partial z}+p_1\frac{\partial Q}{\partial x}\Big)-\Big(\Big(B_{12}+Q-\frac{P_0}{2}\Big)p_1p_2-S_{12}p_2^2\Big)^2\\
&-\frac{1}{K^2}\Big(\rho g-\frac{\partial Q}{\partial z}\Big)p_1\Big(p_1\Big(\frac{\partial B_{12}}{\partial z}+\rho g\Big)+\frac{\partial Q}{\partial x}p_2\Big),
\end{aligned}
$$

$$
\begin{aligned}
S_2 =& -\frac{1}{K}\Big(p_2\frac{3}{5}GK\overline{Y}\mu\Big(B_{11}p_1^2+\Big(Q+\frac{P_0}{2}\Big)p_2^2+S_{12}p_1p_2\Big)\\
&-p_2\frac{3}{10}GK\overline{Y}\mu\Big(B_{22}p_2^2+\Big(Q-\frac{P_0}{2}\Big)p_1^2+S_{12}p_1p_2\Big)\\
&-p_1\Big(\frac{6}{5}GK\overline{Y}\mu-\rho_0g_0\frac{\overline{X}}{\overline{\overline{Z}}}\Big)\Big(\Big(B_{12}+Q-\frac{P_0}{2}\Big)p_1p_2-S_{12}p_2^2\Big)\\
&+\Big(-\rho_0g_0\frac{\overline{X}}{\overline{\overline{Z}}}+\frac{3}{10}GK\overline{Y}\mu\Big)p_1\Big(\Big(B_{12}+Q-\frac{P_0}{2}\Big)p_1p_2-S_{12}p_2^2\Big)\Big).
\end{aligned}
$$

Equating the real and imaginary parts of Eq. (13), we have

$$\rho^2c^4-\rho r_1c^2+S_1=0, \tag{14a}$$

$$\rho r_2c^2+S_2=0. \tag{14b}$$

Thus, from the real part, i.e., Eq. (14a), we have

$$
\begin{aligned}
c_1 &= \Big(\frac{r_1+\sqrt{r_1^2-4S_1}}{2\rho}\Big)^{1/2}=c_{\mathrm{L}},\\
c_2 &= \Big(\frac{r_1-\sqrt{r_1^2-4S_1}}{2\rho}\Big)^{1/2}=c_{\mathrm{T}},
\end{aligned}
$$

where c_{L} is the velocity of the longitudinal wave, and c_{T} is the velocity of the transverse wave.

Thus, P-wave (longitudinal wave) and S-wave (transverse wave) velocities may be expressed in a non-dimensional form as

$$\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}}=\left(\frac{\frac{r_1}{\mu}+\sqrt{(\frac{r_1}{\mu})^2-4\frac{S_1}{\mu^2}}}{6}\right)^{\frac{1}{2}}, \tag{15}$$

$$\frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}}=\left(\frac{\frac{r_1}{\mu}-\sqrt{(\frac{r_1}{\mu})^2-4\frac{S_1}{\mu^2}}}{2}\right)^{\frac{1}{2}}. \tag{16}$$

From the imaginary part, i.e., Eq. (14b), the damping in a dimensionless form is

$$\frac{c_{\mathrm{D}}}{c'_{\mathrm{T}}}=\Big(\frac{-S_2\rho}{\rho r_2\mu}\Big)^{\frac{1}{2}}=\left(\frac{(\frac{-S_2}{\mu^2})}{(\frac{r^2}{\mu})}\right)^{\frac{1}{2}}, \tag{17}$$

where

$$c'_{\mathrm{L}} =\Big(\frac{\lambda+2\mu}{\rho}\Big)^{\frac{1}{2}}, \quad c'_{\mathrm{T}} = \Big(\frac{\mu}{\rho}\Big)^{\frac{1}{2}},$$

$$\frac{r_1}{\mu} =\Big(\frac{B_{11}}{\mu}+\frac{Q}{\mu}+\xi\Big)p_1^2+\Big(\frac{B_{22}}{\mu}+\frac{Q}{\mu}+\xi\Big)p_2^2+4\xi_3 p_1 p_2,$$

$$\begin{aligned}\frac{S_1}{\mu^2} =&\Big(\frac{B_{11}}{\mu}p_1^2+\Big(\frac{Q}{\mu}+\xi\Big)p_2^2+2\xi_3p_1p_2\Big)\Big(\frac{B_{22}}{\mu}p_2^2+\Big(\frac{Q}{\mu}+\xi\Big)p_1^2+2\xi_3p_1p_2\Big)+\frac{9}{50}G^2\overline{Y}^2p_2^2\\ &-\Big(\Big(\frac{B_{12}}{\mu}+\frac{Q}{\mu}-\xi\Big)p_1p_2-2\xi_3p_2^2\Big)^2-\Big(-\frac{G\overline{X}}{\overline{Z}}+\frac{3}{10}G\overline{Y}\Big)\Big(-\frac{G\overline{X}}{\overline{Z}}+\frac{6}{5}G\overline{Y}\Big)p_1^2,\end{aligned}$$

$$\frac{r_2}{\mu} =\frac{3}{10}G\overline{Y}p_2,$$

$$\begin{aligned}\frac{S_2}{\mu^2} =&-\Big(p_2\frac{3}{5}G\overline{Y}\Big(\frac{B_{11}}{\mu}p_1^2+\Big(\frac{Q}{\mu}+\xi\Big)p_2^2+2\xi_3p_1p_2\Big)\\ &-p_2\frac{3}{10}G\overline{Y}\Big(\frac{B_{22}}{\mu}p_2^2+\Big(\frac{Q}{\mu}-\xi\Big)p_1^2+2\xi_3p_1p_2\Big)\\ &-p_1\Big(\frac{6}{5}G\overline{Y}-\frac{G\overline{X}}{\overline{Z}}\Big)\Big(\Big(\frac{B_{12}}{\mu}+\frac{Q}{\mu}-\xi\Big)p_1p_2-2\xi_3p_2^2\Big)\\ &+\Big(-\frac{G\overline{X}}{\overline{Z}}+\frac{3}{10}G\overline{Y}\Big)p_1\Big(\Big(\frac{B_{12}}{\mu}+\frac{Q}{\mu}-\xi\Big)p_1p_2-2\xi_3p_2^2\Big)\Big).\end{aligned}$$

4 Particular cases

Case I The phase velocity equations of P-wave and S-wave propagating in any direction when $\xi=\xi_3=G=0$ are given by

$$\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}}=1, \quad \frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}}=1.$$

In the absence of the initial stress (ξ), the pre-stressed parameter (ξ_3), and Biot's gravity parameter (G), the damping wave does not exist. Moreover, if the ratios $\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}}$ and $\frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}}$ are denoted by $\overline{K}_{\mathrm{L}}$ and $\overline{K}_{\mathrm{T}}$, respectively, i.e., if

$$\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}}=\overline{K}_{\mathrm{L}}, \quad \frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}}=\overline{K}_{\mathrm{T}},$$

then it is clear from Eq. (15) and Eq. (16) that $\overline{K}_{\mathrm{L}}$ and $\overline{K}_{\mathrm{T}}$ are not constants but depend on the initial stress, the pre-stressed parameter, Biot's gravity parameter, the direction of propagation, and the inhomogeneity parameters. $\overline{K}_{\mathrm{L}}$ and $\overline{K}_{\mathrm{T}}$ may be termed as the velocity parameter of an initially stressed medium. In this case, $\overline{K}_{\mathrm{L}}$ and $\overline{K}_{\mathrm{T}}$ being constants equal to unity in the absence of ξ, ξ_3, and G coincide with the results defined by Achenbaeh[5].

Case II When $P_1 = 1$ and $P_2 = 0$, the phase velocities of P-wave and S-wave are given as follows:

$$\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}} = \left(\frac{\frac{r_1}{\mu} + \sqrt{(\frac{r_1}{\mu})^2 - 4\frac{S_1}{\mu^2}}}{6}\right)^{\frac{1}{2}},$$

$$\frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}} = \left(\frac{\frac{r_1}{\mu} - \sqrt{(\frac{r_1}{\mu})^2 - 4\frac{S_1}{\mu^2}}}{2}\right)^{\frac{1}{2}},$$

where

$$\frac{r_1}{\mu} = 4 - \frac{3}{10}GK\overline{X} - \frac{53}{10}\xi + \xi_3,$$

$$\frac{S_1}{\mu^2} = 3\Big(1 - \frac{GK\overline{X}}{5} - \frac{6}{5}\xi\Big)\Big(1 + \frac{3}{10}GK\overline{X} - \frac{7}{10}\xi + \xi_3\Big) - G^2\Big(\frac{3}{10}\overline{Y} - \frac{\overline{X}}{\overline{\overline{Z}}}\Big)\Big(\frac{6}{5}\overline{Y} - \frac{\overline{X}}{\overline{\overline{Z}}}\Big).$$

In this case, the damping wave does not exist as $\frac{r_2}{\mu}$ and $\frac{S_2}{\mu^2}$ vanish.

Case III When $P_1 = 0$ and $P_2 = 1$, the phase velocities of P-wave, S-wave, and the damping wave are given by

$$\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}} = \left(\frac{\frac{r_1}{\mu} + \sqrt{(\frac{r_1}{\mu})^2 - 4\frac{S_1}{\mu^2}}}{6}\right)^{\frac{1}{2}},$$

$$\frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}} = \left(\frac{\frac{r_1}{\mu} - \sqrt{(\frac{r_1}{\mu})^2 - 4\frac{S_1}{\mu^2}}}{2}\right)^{\frac{1}{2}},$$

$$\frac{c_{\mathrm{D}}}{c'_{\mathrm{T}}} = \left(\frac{\left(\frac{-S_2}{\mu^2}\right)}{\left(\frac{r^2}{\mu}\right)}\right)^{\frac{1}{2}},$$

where

$$\frac{r_1}{\mu} = 4 - \frac{3}{10}GK\overline{X} - \frac{17}{10}\xi + \xi_3,$$

$$\frac{S_1}{\mu^2} = \Big(1 + \frac{3}{10}GK\overline{X} + \frac{3}{10}\xi + \xi_3\Big)\Big(3 - \frac{3}{5}GK\overline{X} + \frac{12}{5}\xi\Big) + \frac{9}{50}G^2\overline{Y}^2 - 4\xi_3^2,$$

$$\frac{r_2}{\mu} = -\frac{3}{10}G\overline{Y},$$

$$\frac{S_2}{\mu^2} = -\frac{3}{10}G\overline{Y}(5 + 3\xi + 2\xi_3).$$

Here, it is observed that damping may only exist if $(5+3\xi+2\xi_3) < 0$. Since ξ_3 is always positive, damping wave will only be available if $\xi > 0$, i.e., the initial normal stresses are compressive.

Case IV When $\xi = 0$ and $G = 0$, the phase velocities of P-wave and S-wave are given by

$$\frac{c_{\mathrm{L}}}{c'_{\mathrm{L}}} = \left(\frac{\frac{r_1}{\mu} + \sqrt{(\frac{r_1}{\mu})^2 - 4\frac{S_1}{\mu^2}}}{6}\right)^{\frac{1}{2}}, \quad \frac{c_{\mathrm{T}}}{c'_{\mathrm{T}}} = \left(\frac{\frac{r_1}{\mu} - \sqrt{(\frac{r_1}{\mu})^2 - 4\frac{S_1}{\mu^2}}}{2}\right)^{\frac{1}{2}},$$

where

$$\frac{r_1}{\mu} = 4 + \xi_3 + 2\xi_3 p_1 p_2,$$
$$\frac{S_1}{\mu^2} = (3p_1^2 + p_2^2 + \xi_3 p_2^2 + 2\xi_3 p_1 p_2)(3p_2^2 + p_1^2 + \xi_3 p_1^2 + 2\xi_3 p_1 p_2) - (2p_1 p_2 + \xi_3 p_1 p_2 - 2\xi_3 p_2^2)^2.$$

Here, the damping wave cease to exist as $\frac{r_2}{\mu}$ and $\frac{S_2}{\mu^2}$ are equal to zero.

5 Numerical investigations and discussion

To illustrate the effects of variations of Biot's gravity parameter, the initial stresses, the wave number, and the inhomogeneity parameters on the phase velocity of P-wave and S-wave, numerical calculations are made by Eqs. (15) and (16). The numerical values are taken from Gubbins[18] for various dimensionless material parameters (see Table 1), and graphs are plotted for the dimensionless phase velocities $c_{\mathrm{L}}/c'_{\mathrm{L}}$ and $c_{\mathrm{T}}/c'_{\mathrm{T}}$ against Kz.

Table 1 Values of various dimensionless material parameters

Figures	G	θ	ξ_3	ξ	a/K	b^2/K^2	α/K	β^2/K^2
2	0.02 0.04 0.06	20°	0.2	0.2	2.0	2.5	3.0	1.5
3	0.02	10° 20° 40°	0.2	0.2	2.0	2.5	3.0	1.5
4	0.02	20°	0.0 0.2 0.4	0.2	2.0	2.5	3.0	1.5
5	0.02	20°	0.2	0.0 0.2 0.4	2.0	2.5	3.0	1.5
6	0.02	20°	0.2	0.0 –0.2 –0.4	2.0	2.5	3.0	1.5
7	0.02	20°	0.2	0.2	2.0 2.5 3.0	2.5	3.0	1.5
8	0.02	20°	0.2	0.2	2.0	1.0 1.5 2.0	3.0	1.5
9	0.02	20°	0.2	0.2	2.0	2.5	1.5 2.0 1.5	1.5
10	0.02	20°	0.2	0.2	2.0	2.5	3.0	1.5 2.0 2.5

In Fig. 2, study is made over the effects of Biot's gravity parameter G on the P- and S-wave propagation under the effect of the compressive initial stress $\xi = 0.2$ and the shear stress

$\xi_3 = 0.2$. The values of G for curves 1–2, 3–4, and 5–6 are taken as 0.02, 0.04, and 0.06. It is found that the phase velocity of P-wave always remains to be constant upto $Kz = 0.5$, and then increases remarkably as the wave number increases further. Whereas in the case of S-wave, unlike to P-wave, the phase velocity remains to be constant to certain extent, and then decreases as the wave number increases. It is also noted that at particular wave numbers, i.e., $Kz = 0.9$, 1.1, and 1.25, the phase velocity where P- and S-waves meet may be called as a critical point. As the value of G varies from 0.02 to 0.06, the phase velocity of P-wave increases whereas the phase velocity of S-wave decreases at any particular value of the wave number.

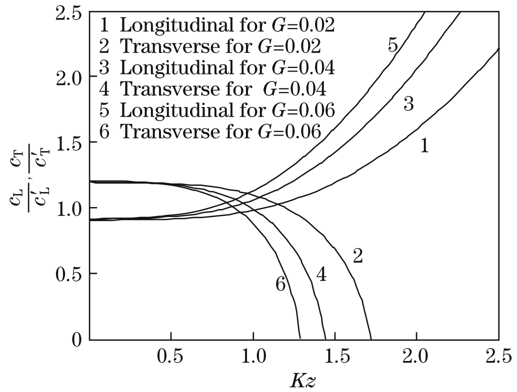

Fig. 2 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $G = 0.02$, 0.04, and 0.06

Figure 3 is plotted to show the effect of the angle of propagation of waves on their phase velocities. Here, study is made at three different angles, i.e., $\theta = 10°$, $20°$, and $40°$. It is found that the P-wave velocity coincides at $Kz = 1.25$, and the sequence of the curves get reversed about this point. The phase velocity of P-wave initially increases with the increase in the angle of propagation, and then after $Kz = 1.25$, it decreases as the angle of propagation increases further. The phase velocity of S-wave increases with the increase in the angle of propagation at a particular wave number. For S-wave, the phase velocity remains to be constant upto a certain value of the wave number, and then it decreases as the wave number increases ahead. This trend shows that as the angle of propagation increases, the velocity remains to be constant for larger wave numbers, and then starts decreasing with the further increase in the wave number.

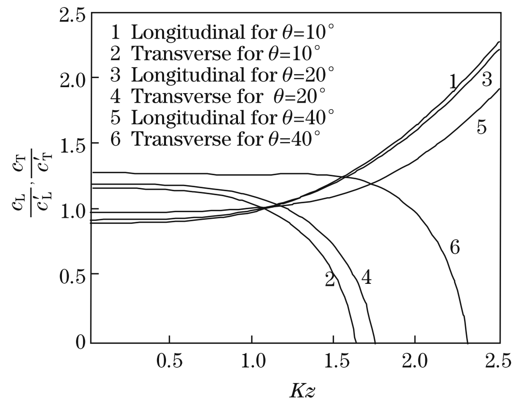

Fig. 3 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $\theta = 10°$, $20°$, $30°$

In Fig. 4, curves are plotted for the phase velocity against the wave number for various values of the pre-stress parameter ξ_3. Curves 1–2, 3–4, and 5–6 are plotted for $\xi_3 = 0.0, 0.2$, and 0.4, respectively. At a particular wave number, as the pre-stress parameter increases, the phase velocity of P-wave increases while the phase velocity of S-wave decreases. It can be observed that in the absence of the pre-stress (curves 1–2), the phase velocity of P-wave is the least while the phase velocity of S-wave is the highest. Curves 2, 4, and 6 being equidistant from each other shows that the pre-stress has a prominent effect on S-wave propagation. Whereas curves 1, 3, and 5 are found to be converging around $Kz = 3.0$, showing that although the varying pre-stress affects the propagation of P-wave initially, the effect diminishes as the curves extend with the further increase in the wave number.

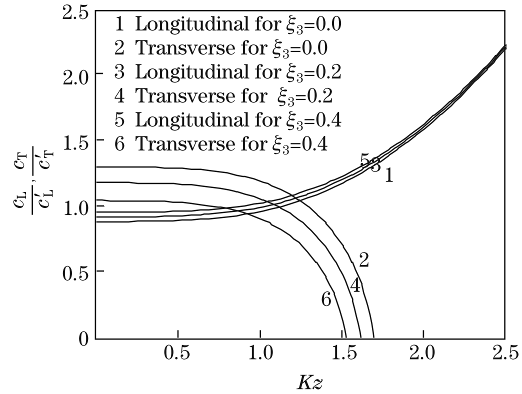

Fig. 4 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $\xi_3 = 0.0$, 0.2, and 0.4

Figures 5–6 involve the effects of the compressive and tensile initial stresses on the phase velocities of P- and S-waves, respectively. The compressive initial stress $\xi = 0.0, 0.2,$ and 0.4 has a dominant effect over the tensile initial stress $\xi = 0.0, -0.2,$ and –0.4. In Fig. 5, the phase velocity of P-wave is found to be decreasing while the phase velocity of S-wave is found to be increasing as the compressive initial stress increases at a particular wave number. Here, curves 1, 3, and 5 may converge at higher wave numbers. In Fig. 6, the tensile initial stress has reverse effects on P- and S-waves, which reflects that it has a promising role in the propagation of longitudinal and transverse waves in the earth's gravitating half space.

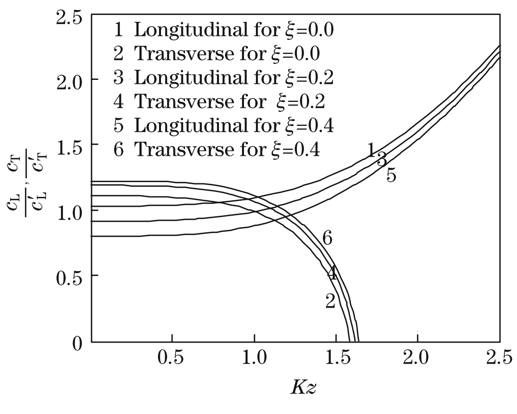

Fig. 5 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $\xi = 0.0$, 0.2, and 0.4

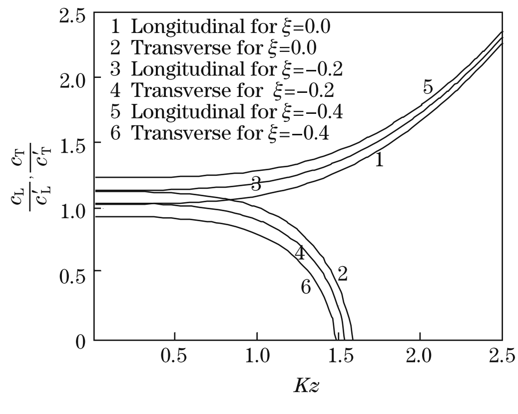

Fig. 6 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $\xi = 0.0$, -0.2, and –0.4

In Figs. 7–8, curves are drawn to explain the effect of the inhomogeneity parameter incorporated in the acceleration due to gravity of the medium. The values of a/K for curves 1–2, 3–4, and 5–6 are taken as 2.0, 2.5, and 3.0, respectively, in Fig. 7, while in Fig. 8, the values of b^2/K^2 for curves 1–2, 3–4, and 5–6 are taken as 1.0, 1.5, and 2.0, respectively. As the value of a/K and b^2/K^2 in Fig. 7 and Fig. 8 increases, the phase velocity of P-wave increases, whereas the phase velocity of S-wave decreases at a particular wave number. The curves drawn in Fig. 7 and Fig. 8 follow the same trend except the few ones as follows:

(i) Compared with the curves in Fig. 8, the curves in Fig. 7 are closer to each other, which shows that although the values of a/K increases by 0.5, the phase velocity increases or decreases to some extent, justifying the fact that the inhomogeneity coefficient a/K has a small bearing on the phase velocity as compared with the inhomogeneity coefficient b^2/K^2 in Fig. 8.

(ii) The curves in Fig. 8 being scattered illustrate the remarkable effect of the variation in b^2/K^2 on the phase velocities of P- and S-waves.

Figures 9–10 display the key role of the inhomogeneity parameter associated with the density

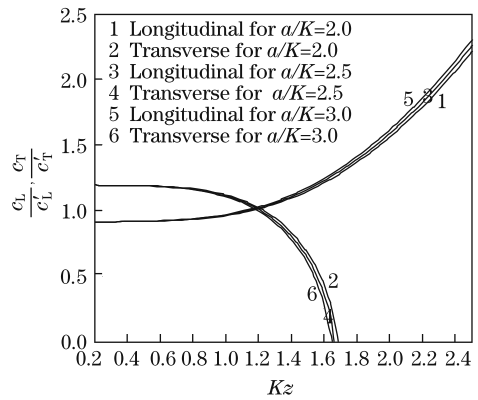

Fig. 7 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $a/K = 2.0$, 2.5, and 3.0

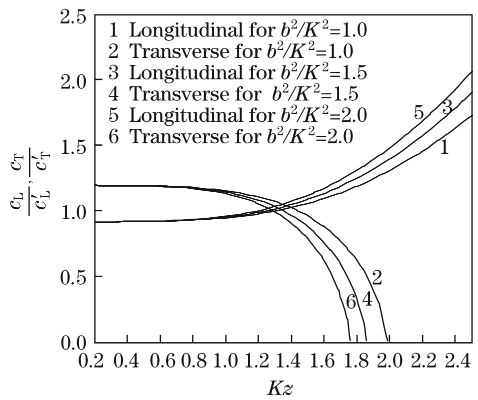

Fig. 8 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $b^2/K^2 = 1.0$, 1.5, and 2.0

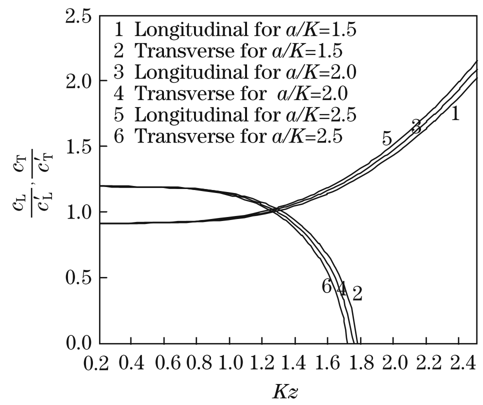

Fig. 9 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $\alpha/K = 1.5$, 2.0, and 2.5

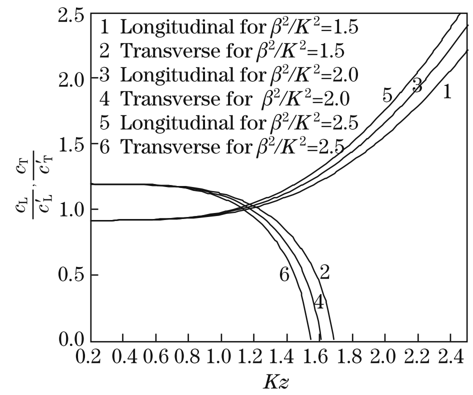

Fig. 10 Dimensionless wave number against dimensionless phase velocities of longitudinal wave and transverse wave for $\beta^2/K^2 = 1.5$, 2.0, and 2.5

of the medium. In Fig. 9, the values of α/K are taken as 1.5, 2.0, and 2.5 for curves 1–2, 3–4, and 5–6, respectively, whereas the values of β^2/K^2 are taken as 1.5, 2.0, and 2.5 in Fig. 10. The phase velocity of P-wave increases while the phase velocity of S-wave decreases as the values of α/K and β^2/K^2 increase for a particular wave number. The curves drawn in Figs. 9–10 have a similar trend as depicted in Figs. 7–8. However, the extremities of the curves in Fig. 10 show that β^2/K^2 has better dominance on the propagation of P- and S-waves.

Hence, the graphs being self explanatory reveal that the effect of Biot's gravity parameter, the angle of propagation, the inhomogeneity parameters, and the wave number on the phase velocities of P-wave and S-wave can be more illustrated using various sets of data in the graphical user interface (GUI) (see Fig. 11) created in MATLAB (version 7.0).

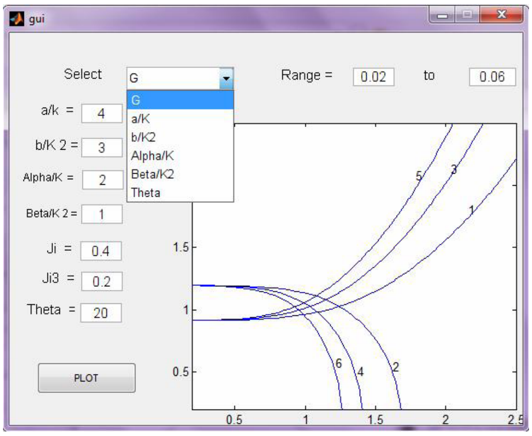

Fig. 11 GUI model developed in MATLAB (version 7.0) showing variation of dimensionless phase velocities of P-wave, S-wave, and damping wave against dimensionless wave number

6 Conclusions

The propagation of P- and S-waves in a half space when the density and gravity vary quadratically under the effect of the initial stress is studied in detail. The dispersion equation is derived in a closed form in the terms of Biot's gravity parameter, the initial stress, the pre-stress parameter, the wave number, and the inhomogeneity parameter associated in the density and the acceleration due to gravity of the half space. The phase velocities are computed numerically, and the effects of each of these parameters are studied by means of graphs. We can observe as follows:

(i) Under the assumed condition, the phase velocity of P-wave increases with the increase in the dimensionless wave number, while the phase velocity of S-wave velocity decreases in all the figures.

(ii) The increasing values of Biot's gravity parameter (G), the pre-stress parameter (ξ_3), and the inhomogeneity parameters a/K, b^2/K^2, α/K, and β^2/K^2 increase the phase velocity of P-wave and decrease the phase velocity of S-wave at a particular wave number. Biot's gravity parameter has a prominent bearing on the phase velocities, while the pre-stress parameter hardly affects the phase velocity of P-wave for higher values of the wave number.

(iii) With the increase in the angle of propagation, when the compressive and tensile stresses increase, the phase velocity of P-wave decreases and the phase velocity of S-wave velocity increases at a particular wave number.

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